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FREE CONVECTION
IN A HEAT-GENERATING FLUID

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Part I. Thermal instability in a horizontal
fluid layer

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Abstract

In the present paper theoretical and experimental studies dealing with the convection of fluids between horizontal surfaces are considered and the effect of heat sources on the onset of a convective motion are discussed.

1. Introduction.

Much attention, both experimental and theoretical, has been given to the convection of fluids between horizontal surfaces at different temperatures (often referred to as the Bénard configuration); when the lower surface is the hotter, instability can bring the fluid into motion. The resulting flow pattern then consists of more or less regularly spaced rising and falling currents and thus one sees an array of "convection cells". The long history of this subject is summarized in Ostrach (1959), Chandrasekhar (1961), Stuart (1963) and Segel (1966).

In these works it has been customary to deal with a quiescent state characterized by a fluid temperature which is decreasing linearly with height. It is of interest to determine in what way the stability would be affected if the quiescent state were characterized by a non-linear temperature profile. Such a non-linear profile could arise if there were an internal heat generation within the fluid. Interest in heat generation fluids stems largely from their nuclear - engineering applications, though such fluids can also be important in chemical engineering and geophysics.

In the present paper the theoretical and experimental studies dealing with the onset of a convective motion for systems with the internal heat generation are discussed.

2. The basic equations.

All analytical work is based on the set of equations, representing the conservation of mass, momentum and energy in a fluid moving under the influence of a body force. Usually this system covers (for a one-component medium):

Navier-Stokes equations of motion:

$$\rho \frac{du_1}{dt} = F_1 \rho - \frac{\partial P}{\partial x_1} + \frac{\partial}{\partial x_1} \left[(\zeta_{tot} - \frac{2}{3}\mu) \operatorname{div} \vec{u} \right] + \frac{\partial}{\partial x_1} \mu \left(\frac{\partial u_1}{\partial x_k} + \frac{\partial u_k}{\partial x_1} \right) \quad (2.1)$$

Here P is the static pressure, ζ_{tot} is the coefficient of volumetric viscosity, μ is dynamic viscosity coefficient, ρ is the density of a medium, F_1 is the projection of mass forces on the x_1 -axis, \vec{u} is the velocity vector, u_1 is its component at the axis, and i, k are indices, according to which there takes place summation at their repetition ($i, k = 1, 2, 3$).

Equation of continuity

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \vec{u} = 0 \quad (2.2)$$

Energy equation

$$\frac{di}{dt} = - \frac{\operatorname{div} \vec{q}}{\rho} + \frac{1}{\rho} \frac{dP}{dt} + \frac{\phi_1}{\rho} + \frac{Q}{\rho} \quad (2.3)$$

where i is the specific enthalpy (per unit mass); Q is the density of distribution of heat sources per unit volume;

$$\phi_1 = \frac{\mu}{2} \left(\frac{\partial u_1}{\partial x_k} + \frac{\partial u_k}{\partial x_1} \right) + (\zeta_{tot} - \frac{2}{3}\mu) (\operatorname{div} \vec{u})^2 = \phi + \zeta_{tot} (\operatorname{div} \vec{u})^2 \quad (2.4)$$

ϕ is dissipative Rayleigh function; \vec{q} is the heat flux density per unit surface per unit time. In the simplest case of a one-component medium when the heat flux to a medium element is determined only by heat conduction, \vec{q} is calculated by Fourier equation

$$\vec{q} = - \lambda \text{ grad } T \quad (2.5)$$

Here T is the medium temperature and λ is the thermal conductivity.

Neglecting pressure terms and the dissipation function in the energy equation and assuming constant fluid properties except in the body force term, the differential equations describing the phenomena are

$$\frac{\partial u_1}{\partial t} + u_k \frac{\partial u_1}{\partial x_k} = - \frac{1}{\rho_0} \frac{\partial p}{\partial x_1} - \frac{\rho g}{\rho_0} \delta_{13} + \nu \nabla^2 u_1 \quad (2.6)$$

$$\frac{\partial u_1}{\partial x_1} = 0 \quad (2.7)$$

$$\frac{\partial T}{\partial t} + u_k \frac{\partial T}{\partial x_k} = \kappa \nabla^2 T \quad (2.8)$$

where g is the acceleration of gravity, ν is the kinematic viscosity, κ is the thermal diffusivity, δ_{1j} is the Kronecker delta and ∇^2 is the Laplacian.

Equations (2.6) - (2.8) are the basic equations in the Boussinesq approximation. They must be supplemented by an equation of state. For substances with which we shall be principally concerned, we can write

$$\rho = \rho_0 [1 - \beta(T - T_0)] \quad (2.9)$$

where β is the coefficient of volume expansion and T_0 is the standard temperature at which $\rho = \rho_0$.

3. Thermal instability in a horizontal fluid layer.

Consider the stability of an initially quiescent horizontal fluid layer which supports a temperature gradient in the vertical. The gradient is included by a prescribed temperature difference ΔT (heated below) across the layer plus a uniform distribution of heat sources of intensity Q in the fluid.

The steady temperature of the quiescent state is given by

$$T - T_0 = - \frac{Q}{2\lambda}(x_3^2 - \frac{h^2}{4}) - A \cdot x_3, \quad (3.1)$$

where $A = \frac{\Delta T}{h}$ with h denoting the depth of layer, and T_0 is the arithmetic mean of the boundary temperatures. It is also possible to regard (3.1) as the solution of the conduction equation for a fixed value ΔT and a mean temperature $T_0 = \frac{Qkt}{\lambda}$ which increases linearly in the time t (Krishnamurti 1968). For this situation a solution to the conduction equation,

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x_3^2},$$

can be separated into $T = T_0 + T_s(x_3)$. Integration of the conduction equation for $T_s(x_3)$ then gives (3.1) written for $T \rightarrow T_s$, $T_0 \rightarrow T_{0s}$ and $\Delta T_s = \Delta T$. It then follows that $T(t, x_3)$ is also represented by (3.1).

The stability of the conduction solution can be treated within the framework of the Boussinesq equations (2.6) - (2.8). The problem, in one of its many forms, has been treated by Sparrow, Goldstein and Jonson (1964), Joseph and Shir (1966), Debler (1966), Roberts (1967), Krishnamurti (1968) and Joseph, Goldstein and Graham (1968).

The temperature may be written

$$T = T_s + \theta \quad (3.2)$$

To get a dimensionless form of the equations we set

$$\begin{aligned} x_1 &= hx'_1, & u_1 &= \kappa u'_1/h, & t &= h^2 t'/\kappa, \\ \theta &= \kappa \nu \theta' / \beta g h^3, & P &= \kappa^2 \rho_0 p' / h^2. \end{aligned}$$

Disregarding the static pressure, applying (2.7) and (3.2), and dropping the primes, we obtain

$$\frac{\partial u_1}{\partial t} + u_k \frac{\partial u_1}{\partial x_k} = - \frac{\partial p}{\partial x_1} + P \theta \delta_{13} + P \nabla^2 u_1 \quad (3.3)$$

$$\frac{\partial u_1}{\partial x_1} = 0 \quad (3.4)$$

$$\frac{\partial \theta}{\partial t} + u_k \frac{\partial \theta}{\partial x_k} = \nabla^2 \theta + [R + \bar{Q} \cdot (x_3 - \frac{1}{2})] \cdot u_3 \quad (3.5)$$

Here P is the Prandtl number, R is the Rayleigh number and \bar{Q} is the heat source parameter:

$$P = \frac{\nu}{\kappa}, \quad R = \frac{Ag \beta h^4}{\nu \kappa}, \quad \bar{Q} = \frac{g \beta Q h^5}{\nu \kappa^2} \quad (3.6)$$

The horizontal boundaries may be either rigid or free. In the first case $u_1 = 0$ at the boundary; In the last case the vertical velocity and the shearing stresses are zero at the boundary. It will furthermore be assumed that the horizontal boundaries are either perfect heat conductors or perfect heat insulators. Applying (3.4) we then have

$$u_1 = 0, \quad \theta = 0 \quad \text{or} \quad \frac{\partial \theta}{\partial n} = 0 \quad \text{at rigid boundaries} \quad (3.7)$$

$$u_3 = \frac{\partial^2 u_3}{\partial x_3^2} = 0, \quad \theta = 0 \quad \text{or} \quad \frac{\partial \theta}{\partial n} = 0 \quad \text{at free boundaries.}$$

The boundary-value problem (3.3) - (3.5) and (3.7) governs the difference between the altered motion and the conduction solution. There are several ways to make deductions about this postulated motion.

a) Linear theory. The linearized version of equations (3.3)-(3.5) together with the proper boundary conditions lead to an eigenvalue problem which determines the critical Rayleigh number, R , corresponding to the onset of convection. It is easily shown that this problem is selfadjoint. In the linear eigenvalue problem $\partial/\partial t$ operator may therefore be cancelled. Eliminating u_1 and u_2 and applying (3.4) the linearized equations may be written:

$$\nabla^4 u_3 + \nabla^2 \theta = 0 \quad (3.8)$$

$$\left[R + \overline{Q}(x_3 - \frac{1}{2}) \right]$$

where ∇_2^2 is the two-dimensional Laplacian. The theory derived on the basis of such linearized equations is called the linear stability theory in contrast to non-linear theories which attempt to allow for the finite amplitudes of the perturbations.

The general solution of (3.8) may be written (see for example Chandrasekhar 1961, chapter 2)

$$\begin{aligned} u_3 &= f(x_3) \cdot F(x_1, x_2) \\ \theta &= g(x_3) \cdot F(x_1, x_2) \end{aligned} \quad (3.9)$$

Here $F(x_1, x_2)$ satisfies

$$\frac{\partial^2 F}{\partial x_1^2} + \frac{\partial^2 F}{\partial x_2^2} = -a^2 F \quad (3.10)$$

The wave-number a decides the horizontal scale of the convective motions; the type of solution to (3.10) which is selected decides their horizontal structure, or 'planform'.

It may be shown that, irrespective of the solution to (3.10) which is selected, we have

$$\left(\frac{d^2}{dx_3^2} - a^2\right)^2 f(x_3) - a^2 g(x_3) = 0 \quad (3.11)$$

$$\left[R + \bar{Q}(x_3 - \frac{1}{2})\right] f(x_3) + \left(\frac{d^2}{dx_3^2} - a^2\right) g(x_3) = 0$$

The problem of solving (3.11) for a fluid layer with rigid surfaces heated internally and from below has been considered by Sparrow, Goldstein and Jonson (1965). Joseph, Goldstein and Graham (1968) have considered a similar problem for a fluid layer with free surfaces. The critical numbers which have been computed in this way are plotted in figure 1 for parametric values $\frac{1}{2} \frac{\bar{Q}}{R}$.

The marginal stability problem for a layer of fluid, bounded above by a rigid plate held at constant temperature, and below by thermal insulator was calculated by Roberts (1967), his values being $\bar{Q} = 2772$ and $a = 2.63$ respectively. It had been previously estimated by Debler (1966) as 2786 and 2.5.

Linear theory gives conditions (a critical Rayleigh number) under which hydrodynamic systems are definitely unstable.

b) Energy theory. The energy method judges stability or instability of a given fluid motion by whether the energy of a disturbance of the given motion grows or decays. If the values of certain stability parameters are below critical values, the energy decreases and the hydrodynamic system is called stable. Energy methods frequently enable rigorous stability deductions to be made. An intrinsic deficiency of energy methods is that they give conditions guaranteeing stability even against disturbances not satisfying the equations of motion. An understanding of the effect of large physically allowable disturbances is just beginning to emerge.

In the modern theory one considers the global energy of a difference motion. The global energy, kinematic conditions and boundary constraints are used in two lines of deduction. The first of these leads to a universal stability criterion, universal in the sense that specific details of the basic motion and details of the flow geometry need not be completely specified. A second line of deduction leads to the formulation of a maximum problem and achieves a sharper result by making more efficient use of known details of the basic flow. The procedure is developed in Joseph's (1966) paper.

The essential elements of the energy method as this is applied to Boussinesq fluids evolve from deductions made from the energy identifies

$$\frac{dK}{dt} = \frac{d}{dt} \int \frac{1}{2} a^2 \quad (3.12)$$

and

$$P \frac{d\theta}{dt} = P \frac{d}{dt} \int \frac{1}{2} \theta \quad (3.13)$$

where κ is the kinetic energy, and θ the temperature modulus, scales the magnitude of changes in internal (thermal) energy. Equations for κ and θ follow from the integration of suitably multiplied differential equations (2.6) and (2.8).

Joseph's problem is to determine the 'best' value of the positive parameter λ for which the inequality

$$\frac{d}{dt}(\kappa + \lambda P\theta) \leq 0 \quad (3.14)$$

holds for all $t > 0$. This criterion allows for the possibility that either κ or θ may momentarily increase while the sum $(\kappa + \lambda P\theta)$ decreases monotonically in time. It is obvious that $(\kappa + \lambda P\theta)$ plays the role of an energy for the system and (3.14) is an energy criteria.

The problem is formulated in the framework of variational calculus. The energy functional and associated Euler equations for the system are formed by Joseph. The Euler equations are

$$\frac{1}{2}R_\lambda \left[\lambda \frac{\bar{Q}(1-2x_3)-2R}{\bar{Q}+2R} - 1 \right] u_3 = \lambda \nabla^2 \theta \quad (3.15)$$

and

$$\frac{1}{2}R_\lambda \left[\lambda \frac{\bar{Q}(1-2x_3)-2R}{\bar{Q}+2R} - 1 \right] \theta \delta_{13} = - \frac{\partial P}{\partial x_1} + \nabla^2 u_1 \quad (3.16)$$

subject to (2.7) and (3.7). Here R_λ is the Lagrange multiplier. Stability is guaranteed when the square root of the Rayleigh number is less than the smallest positive eigenvalue of (3.15) and (3.16) for fixed $\frac{\bar{Q}}{R}$ and any $\lambda > 0$. The largest of these smallest values $R_\lambda(\lambda, \frac{\bar{Q}}{R})$ is the energy limit

$$\bar{R}(\frac{\bar{Q}}{\bar{R}}) = \max_{\lambda > 0} R_{\lambda} \quad (3.17)$$

The quiescent state is stable to disturbances of any magnitude provided that

$$R < \bar{R}^2 \quad (3.18)$$

The value

$$\lambda = - \int_0^1 dz u_3 \theta \delta_{1s} \cdot \int_0^1 dz \left[\frac{\bar{Q}(1-2x_3)-2R}{\bar{Q}+2R} \right] u_3 \theta \delta_{1s} \quad (3.19)$$

which makes R_{λ} maximum evidently reduces to unity for $\bar{Q} = 0$. For this case energy and linear theory coincide, exchange of stability also applies, and the conduction solution is subcritically stable. Energy theory gives conditions under which hydrodynamic systems are definitely stable. It cannot with certainty conclude instability. Comparison of the stability limits as given by energy and linear theory yields the range of values of relevant stability parameters in which subcritical instabilities of the hydrodynamic system are possible.

c) Perturbation analysis.

Krishnamurti has given a perturbation analysis for this problem. Using a double perturbation series in the amplitude of the motion and $\frac{\bar{Q}}{\bar{R}}$, detailed finite amplitude results are obtained. Subcritical flow is present only for hexagonal planforms and this flow is stable. Subcritical motion can persist for values of the Rayleigh number $R_{\min} \leq R < R_c$ where

$$R_{\min} - R_c = - \frac{\bar{Q}^2}{9\pi R^2} \frac{\{5.92^2 + 2(5.92)(1.1838)/P + (1.1838)^2/P^2\}}{[6.89 + (0.0457)/P + (0.0709)/P^2]}$$

The nature of the perturbation series as well as the possible existence of other subcritical solutions have yet to be established. All subcritical solutions, of whatever form, must be within the shaded region of Fig. 1.

The fact that subcritical instabilities do actually occur in this range deemed open by energy theory is confirmed by the experiments of Krishnamurti.

The curve II in figure 2 is one for which the temperature difference was slowly increased with the mean temperature changing at the rate $3.6^\circ/\text{hour}$. The critical Rayleigh number predicted for this rate is 1465, which is 14% below the critical number for $\bar{Q} = 0$. The observed critical temperature difference is, however, about 40% below the critical point for $\bar{Q} = 0$. This is interpreted as a finite amplitude instability occurring at a Rayleigh number below that predicted by linear theory.

4. Summary.

With increasing departures from the linear temperature profile (i.e. with increasing heat-source intensity), it is found that the fluid layer becomes more prone to instability, that is the critical Rayleigh number decreases. Rayleigh numbers calculated from linear and energy theories do not coincide when internal heat sources are present. The linear analysis gives values of the Rayleigh number above which infinitesimally small disturbances will be amplified. The stability boundary, from the energy theory, gives Rayleigh numbers below which even large disturbances will not be amplified. All subcritical solutions, of whatever form, must be within this region.

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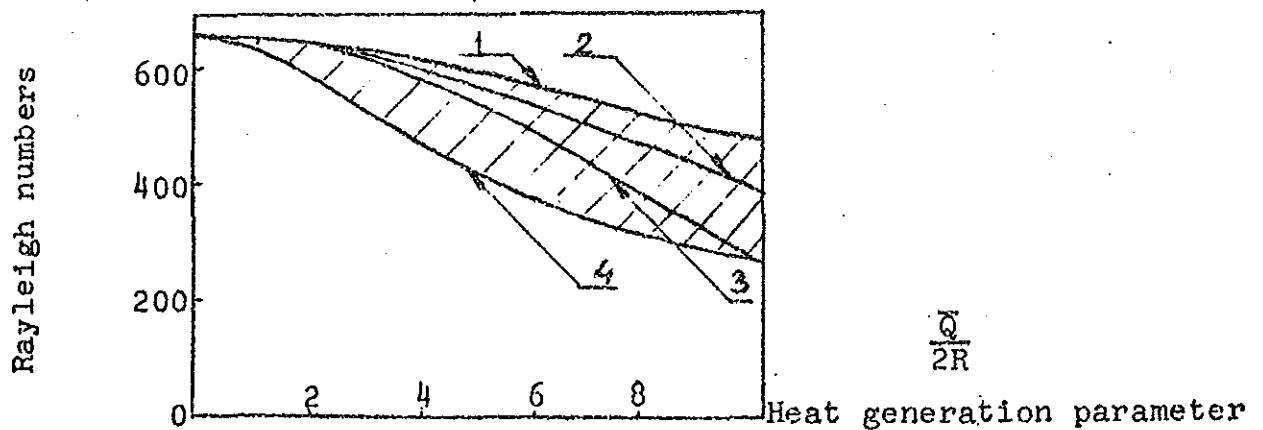


Figure 1 Regions of stability and instability for a fluid layer with free surfaces heated from below and internally.

1. Instability boundary, R_c
2. Perturbation analysis, R_{\min} for $P = \infty$
3. Perturbation analysis, R_{\min} for $P = 0$
4. Stability boundary, R^2

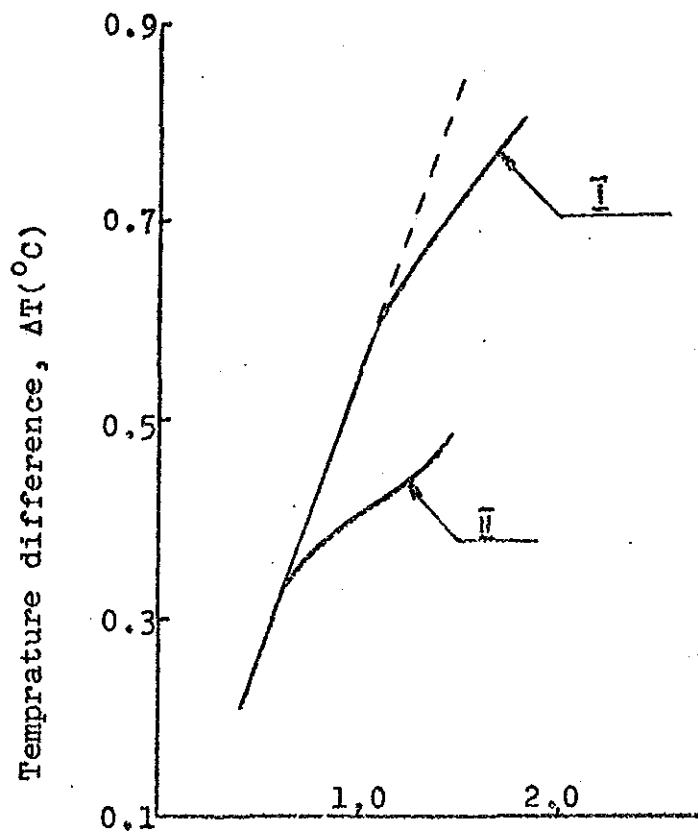


Figure 2 Heat flux plotted against temperature difference
(From Krishnamurti)

References

- Chandrasekhar, S. 1961 Hydrodynamic and Hydromagnetic
Stability, Oxford University Press
- Debler, W.R. 1966 J. Fluid Mech. 24, 165
- Joseph, D. and Shir, C.C. 1966 J. Fluid Mech. 26, 753
- Joseph, D., Goldstein R. and
Graham D. 1968 Phys. Fluids, 11, 903
- Krishnamurti, R. 1968 J. Fluid Mech. 33, 445
- Ostrach, S. 1957 Trans.Am.Soc.Mech.Engrs. 79, 299
- Roberts, P.H. 1967 J. Fluid Mech. 30, 33
- Segel, L.A. 1966 Non-linear Problems in Hydrodynamic
Stability. In Non-equilibrium thermo-
dynamics, variational techniques
and stability. Eds. Donelly, R.J.,
herman , R. & Prigogine, I.
- Sparrow, E.M., Goldstein, R.J.
and Jonsson, V.K. 1964 J. Fluid Mech. 18, 513
- Stuart, J.T. 1963 Hydrodynamic Stability, Sect. IX of
Laminar Boundary Layers ed. by
L. Rosenhead, Oxford University Press.